In a nutshell: Euler's method

Given the initial-value problem (IVP)

$$y^{(1)}(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

we would like to approximate the solution $y(t)$. This algorithm uses Taylor series and iteration. We are given a step size $h > 0$ and a maximum number of steps N and we define $t_k = t_0 + hk$. We want to approximate the solution on the interval $[t_0, t_N]$.

Given an approximation at a point t_k where $y(t_k) \approx y_k$, we will find the approximation y_{k+1} which approximates the solution at $t_k + h = t_{k+1}$.

1. Let $k \leftarrow 0$.
2. If $k = N$, we are finished: we have approximated $y(t_k)$ for $k = 1, \ldots, N$.
3. Let $y_{k+1} \leftarrow y_k + h\,f(t_k, y_k)$.
4. Increment k and return to Step 2.

Error analysis

For a single step, Euler's method is $O(h^2)$, for assuming y_k is exact,

$$\begin{aligned} y(t_{k+1}) = y(t_k + h) &= y(t_k) + y^{(1)}(t_k)h + \frac{1}{2}y^{(2)}(\tau)h^2 \\ &= y_k + hf(t_k, y_k) + \frac{1}{2}y^{(2)}(\tau)h^2 \end{aligned}$$

However, over multiple steps, where we are using approximations to estimate the next approximation, the error reduces to $O(h)$.